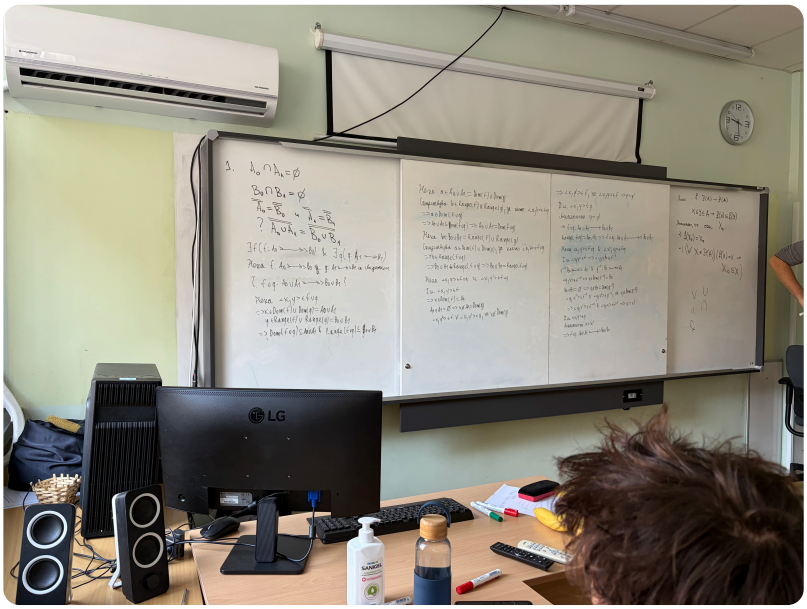
Set Theory — 06.11.2026

Решения на домашното

1. *Твърдение* Нека $A_1 \cap A_2 = \varnothing$, $B_1 \cap B_2 = \varnothing$, $\overline{\overline{A_0}} = \overline{\overline{B_0}}$ и $\overline{\overline{A_1}} = \overline{\overline{B_1}}$. ? $\overline{\overline{A_0 \cup B_0}} = \overline{\overline{A_1 \cup B_1}}$?
   Доказателство на Николайчо:
   Съществува инекция f от A_0 в B_0 и инекция g от A_1 в B_1.
   

2. *Твърдение* Ако $\overline{\overline{A}} < \overline{\overline{B}}$ и $\overline{\overline{B}} < \overline{\overline{C}}$, то $\overline{\overline{A}} < \overline{\overline{C}}$
   Доказателство на Кочев:
   $\overline{\overline{A}} < \overline{\overline{B}} \Rightarrow \overline{\overline{A}} \neq \overline{\overline{B}}$ & $\overline{\overline{A}} \leq \overline{\overline{B}}$
   $\overline{\overline{B}} < \overline{\overline{C}} \Rightarrow \overline{\overline{B}} \neq \overline{\overline{C}}$ & $\overline{\overline{B}} \leq \overline{\overline{C}}$
   $\Rightarrow \overline{\overline{A}} \leq \overline{\overline{C}}$
   Ако $\overline{\overline{A}} = \overline{\overline{C}}$, съществува $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow C \Rightarrow f^{-1} : C \rightarrowtail\!\!\!\!\twoheadrightarrow A$
   Съществува $g : B \rightarrowtail C$
   $\Rightarrow f^{-1} \circ g : B \rightarrowtail A \Rightarrow \overline{\overline{B}} \leq \overline{\overline{A}} \Rightarrow \overline{\overline{A}} = \overline{\overline{B}}$ Absurd!
   $\Rightarrow \overline{\overline{A}} \neq \overline{\overline{C}}$
   $\Rightarrow \overline{\overline{A}} < \overline{\overline{C}}$
3. *Твърдение* $A \subseteq B \subseteq C$ и $\overline{\overline{A}} = \overline{\overline{C}}$, то $\overline{\overline{A}} = \overline{\overline{B}}$
   Доказателство на Кочев:
   $A \subseteq B$
   $\Rightarrow id_A : A \rightarrowtail B$
   $\Rightarrow \overline{\overline{A}} \leq \overline{\overline{B}}$

$B \subseteq C$
$\Rightarrow id_B : B \rightarrowtail C$
$\Rightarrow \overline{\overline{B}} \leq \overline{\overline{C}} = \overline{\overline{A}}$

$\Rightarrow \overline{\overline{B}} \leq \overline{\overline{A}}$

$\Rightarrow \overline{\overline{A}} = \overline{\overline{B}}$

не домашно

$\overline{\overline{\text{чертичките}}}$ са някаква операция.

Въпрос Какво изискване да поставим върху *A*, така че да съществува множество C, такова че $x \in C \leftrightarrow \overline{\overline{x}} = \overline{\overline{A}}$?

Има такова множество — празното.

Твърдение Съществува множество $C : x \in C \leftrightarrow \overline{\overline{x}} = \overline{\overline{A}}$ тогава и само тогава когато $A = \varnothing$.

$\leftarrow$) $\{\varnothing\}$ е свидетел

$\rightarrow$) Нека $A \neq \varnothing$. Нека C е множество, такова че $\forall x (x \in C \leftrightarrow \overline{\overline{x}} = \overline{\overline{A}})$.

Нека p е произволно множество.

1 сл. $p \in A$. $p \in \cup C$ (защото $A \in C$)

2 сл. $p \notin A$.

$A \neq \varnothing$. Нека $q \in A$ е свидетел.

? $\overline{\overline{\Big(A \setminus \{q\}\Big) \cup \{p\}}} = \overline{\overline{A}}$?

$id_{A \setminus \{q\}} \cup \{\langle q, p\rangle\} : \Big(A \setminus \{q\}\Big) \cup \{p\} \rightarrowtail\!\!\!\!\!\twoheadrightarrow A.$

$p \in \Big(A \setminus \{q\}\Big) \cup \{p\} \in C$. $p \in \cup C$ Absurd!

Втори начин

Нека $A \neq \varnothing$. Нека p — произволно.

Нека C е множество: $x \in C \leftrightarrow \overline{\overline{x}} = \overline{\overline{A}}$.

$\overline{\overline{\{p\} \times A}} = \overline{\overline{A}}$ — оцветявам A в цвета p.

Елементите на това нещо са $(\forall a \in A)\Big(\langle p, a\rangle \in \{p\} \times A\Big)$

$f(a) \leftrightharpoons \langle p, a\rangle$

$\Big(\{p\} \times A\Big) \in C.$

$A \neq \varnothing$. Съществува $q \in A$

$\langle p, q\rangle \in \{p\} \times A \in C$

$\langle p, q\rangle \in \cup C$

$\{p\} \in \Big\{\{p\}, \{p, q\}\Big\} \in \cup C$

$\{p\} \in \cup\cup C$, $p \in \cup\cup\cup C$. Absurd!

$A_0 \cap A_1 \neq \varnothing$

$\Big(\{0\} \times A_0\Big) \cap \Big(\{1\} \times A_1\Big) = \varnothing$

Проекциите на функциите

Твърдение Ако $\overline{\overline{A_0}} = \overline{\overline{A_1}}$ и $\overline{\overline{B_0}} = \overline{\overline{B_1}}$, то $\overline{\overline{A_0 \times B_0}} = \overline{\overline{A_1 \times B_1}}$

Има $f : A_0 \rightarrowtail\!\!\!\!\twoheadrightarrow B_0$ и $g : A_1 \rightarrowtail\!\!\!\!\twoheadrightarrow B_1$

$\langle a_0, b_0 \rangle \in A_0 \times B_0$

$\langle f(a_0), g(b_0) \rangle \in A_1 \times B_1$

$h \subseteq (A_0 \times B_0) \times (A_1 \times B_1)$

$\big\langle \langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle \big\rangle \in h \leftrightarrow a_1 = f(a_0)\ \&\ b_1 = g(b_0)$

Нека $a' \in A_0$ и $b' \in B_0$

$\langle a', b' \rangle \in \mathrm{Dom}(h)$

$\big\langle \langle a', b' \rangle, \langle f(a'), f(b') \rangle \big\rangle \in h.$

Нека $a'' \in A_1$ и $b'' \in B_1$.

$\langle a'', b'' \rangle \in \mathrm{Range}(h)$, защото $\big\langle \langle f^{-1}(a''), g^{-1}(b'') \rangle, \langle a'', b'' \rangle \big\rangle \in h.$

Ако $\langle x, y \rangle \in h$ и $\langle x, y' \rangle \in h$.

$x = \langle a, b \rangle$, $a \in A_0$ и $b \in B_0$

$y = \langle a', b' \rangle$, $a' \in A_1$ и $b' \in B_1$

$y' = \langle a'', b'' \rangle$, $a'' \in A_1$ и $b'' \in B_1$

$a' = f(a)\ b' = g(b)$

$a'' = f(a)\ b'' = g(b)$

$a' = a''\ b' = b''$

$y = y'.$

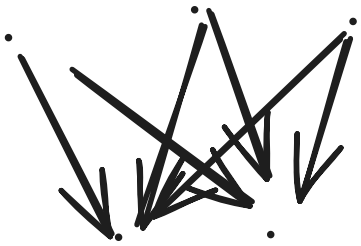
Дефиниция Нека A и B са множества. С B^A бележим множеството от всички функции от A към B.

Съществува ли? Ако $f : A \to B$, то $f \subseteq A \times B$.

$f \in \mathcal{P}(A \times B)$

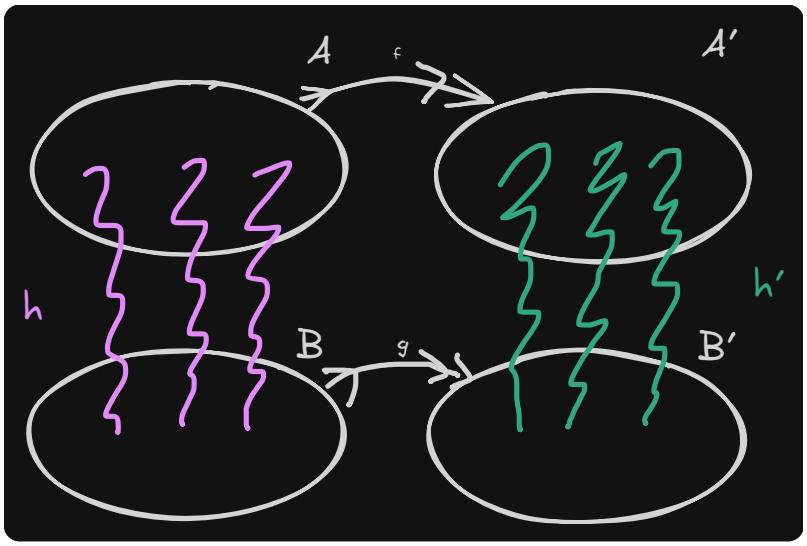
Схема за отделяне: $B^A \leftrightharpoons \Big\{ f \mid f \in \mathcal{P}(A \times B)\ \&\ f : A \to B \Big\}$

2^3



Твърдение Ако $\overline{\overline{A}} = \overline{\overline{A'}}$ и $\overline{\overline{B}} = \overline{\overline{B'}}$, то $\overline{\overline{B^A}} = \overline{\overline{B'^{A'}}}$

Доказателство: Нека $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow A'$, $g : B \rightarrowtail\!\!\!\!\twoheadrightarrow B'$



Нека $h \in B^A (h : A \to B)$
? $h' : A' \to B'$?
Ако $a' \in A$
$h'(a') = g(h\Big(f^{-1}(a')\Big)) \in B'$
$h' : A' \to B'$. $h' \in {B'}^{A'}$
$i \subseteq B^A \times {B'}^{A'}$
$\langle h, h' \rangle \in i \leftrightarrow (\forall a' \in A)\Big(h'(a') = g(h\Big(f^{-1}(a')\Big))\Big)$
? $i : B^A \rightarrowtail\!\!\!\!\!\rightarrow {B'}^{A'}$?
Нека $h \in B^A$. Нека от $A' \times B'$ отделим наредените двойки $\langle a', g(h(f^{-1}(a'))) \rangle$ за $a' \in A'$ и ги кръстим h'
$h' : A' \to B'$, $\langle h, h' \rangle \in i$
$h \in \mathrm{Dom}(i)$
Нека $h' \in {B'}^{A'}$
Отделям от $A \times B$ наредените двойки $\langle a, g^{-1}(h'(f(a))) \rangle$ за $a \in A$ в множеството h.
$\langle h, h' \rangle \in i$. $h' \in \mathrm{Range}(i)$.
$\mathrm{Dom}(i) B^A$, $\mathrm{Range}(i) = {B'}^{A'}$.
Нека $\langle h, h' \rangle \in i$ и $\langle h, h'' \rangle \in i$
Нека $\langle a', b' \rangle \in h'$ $(a' \in A', b' \in B')$
$b' = h'(a')$. Понеже $\langle h, h' \rangle \in i$.
$b' = g(h(f^{-1}(a')))$
Понеже $\langle h, h'' \rangle \in i$, $h''(a') = g(h(f^{-1}(a')))$
$\langle a, h''(a') \rangle \in h''$ (защото $a' \in A'$)
$h''(a') = b$
$\langle a', b' \rangle \in h''$
$h' \subseteq h''$
Аналогично $h'' \subseteq h'$. ? Тоест $h' = h''$
$i : B^A \twoheadrightarrow {B'}^{A'}$
Нека h, $h'' \in B^A$, $h \in {B'}^{A'}$
$i(h') = h$ $i(h'') = h'$
Нека $\langle a, b \rangle \in h'$. ? $\langle a, b \rangle \in h''$?
$g^{-1}(h(f(a))) = b$, защото $h = i(h')$
$i(h')(f(a)) = g(h'(f^{-1}(f(a)))) = g(h'(a))$

$g^{-1}(h(f(a))) = g^{-1}(i(h')(f(a))) = g^{-1}(g(h'(a))) = h'(a)$
$h'(a) = g^{-1}(h(f(a)))$
$h = i(h'')$
$i(h'')(f(a)) = g(h''(f^{-1}(f(a)))) = g(h''(a))$.
$g^{-1}(h(f(a))) = g^{-1}(g(h''(a))) = h''(a)$
$h'(a) = h''(a)$ за произволно a.
$\langle a, b\rangle \in h''$ $h' \subseteq h''$
Нека $\langle a, b\rangle \in h''$
$h'(a) = h''(a)$
$\langle a, b\rangle \in h''$, тъй като $h'(a) = h''(a)$, както доказахме за $a \in A$.
$h'' \subseteq h'$ $h' = h''$
$i : B^A \rightarrowtail\!\!\!\!\twoheadrightarrow B'^{A'}$

Твърдение за десет минути

Твърдение $\overline{\overline{2^A}} = \overline{\overline{\mathcal{P}(A)}}$
Нека да строим функция $h : \mathcal{P}(A) \rightarrowtail\!\!\!\!\twoheadrightarrow 2^A$
Нека $X \subseteq A$ ($X \in \mathcal{P}(A)$)
$f_X^A \leftrightharpoons (X \times \{1\}) \cup \Big((A \setminus X) \times \{0\}\Big)$
$f_X^A(x) = 0$, ако $x \notin X$; 1, ако $x \in X$.
f_X^A се нарича характеристична функция на X, като подмножество на A
$h(X) \leftrightharpoons f_X^A$
Нека $g : A \to 2$.
Съществува ли $x \subseteq A$, такова че $h(X) = g$?
Нека $X_0 \leftrightharpoons \{x \mid x \in A \ \&\ g(x) = 1\}$
$h(X_0) = f_{X_0}^A$. $f_{X_0}^A$?$=g$
$f_{X_0}^A = \Big(X_0 \times \{1\}\Big) \cup \Big((A \setminus X_0) \times \{0\}\Big)$=
$= \Big(\{X \mid x \in A \ \&\ g(x) = 1\} \times \{1\}\Big) \cup \Big((A \setminus \{x \in A \ \&\ g(x) = 1\}) \times \{0\}\Big)$
Нека $x \in A$. $g(x)$?$=f_{X_0}^A(x)$
1 сл. $g(x) = 0$. $x \notin X_0$
$\langle x, 0\rangle \in f_{X_0}^A$, такива че $f_{X_0}^A = 0$
2 сл. $g(x) = 1$. $x \in X_0$
$\langle x, 1\rangle \in f_{X_0}^A$, такива че $f_{X_0}^A(x) = 1$

$g = f_{X_0}^A$
Нека $X' \subseteq A$, $X'' \subseteq A$
$f_{X'}^A = f_{X''}^A$
$L \leftrightharpoons (X' \times \{1\}) \cup ((A \setminus X') \times \{0\}) = (X'' \times \{1\}) \cup ((A \setminus X'') \times \{0\}) \leftrightharpoons R$
Нека $x \in X'$. $\langle x, 1\rangle \in L$ значи $\langle x, \rangle 1 \in X'' \times \{1\}$, $x \in X''$, $X' \subseteq X''$.
Аналогично $X'' \subseteq X'$.
$h : \mathcal{P}(A) \rightarrowtail\!\!\!\!\twoheadrightarrow 2^A$

Домашно

1. $\overline{\overline{A \times B}} = \overline{\overline{B \times A}}$
2. $\overline{\overline{(A \times B) \times C}} = \overline{\overline{A \times (B \times C)}}$
3. $\overline{\overline{A^B \times A^C}} = \overline{\overline{A^{B \cup C}}}$, ако $B \cap C = \varnothing$
4. Нека $a^* \notin A$ и $b^* \notin \mathcal{P}(A)$ и $\overline{\overline{A = A \cup \{a^*\}}}$. Докажете $\overline{\overline{\mathcal{P}(A)}} = \overline{\overline{\mathcal{P}(A) \cup \{b^*\}}}$
   Дефиниция на девикин за безкрайност