Set Theory — 30.10.2026

Домашно решения

Ако $f : A \rightarrowtail B$ и $g : B \rightarrowtail C$, то $g \circ f : A \rightarrowtail C$

Доказателство Допускаме противното: $\exists x \exists x' \left((g \circ f)(x) = a \ \& \ (g \circ f)(x') = a \ \& \ x \neq x'\right)$
$(g \circ f)(x) = a \ \& \ (g \circ f)(x') \leftrightharpoons g(f(x)) = a \ \& \ g(f(x')) = a$
$g : B \rightarrowtail C$ (инекция)
$g(f(x)) = g(f(x')) \Rightarrow f(x) = f(x')$
$f : A \rightarrowtail B$ (инекция)
$f(x) = f(x') \Rightarrow x = x'$ Absurd!

Ако $f : A \twoheadrightarrow B$ и $g : B \twoheadrightarrow C$, то $g \circ f : A \twoheadrightarrow C$

Доказателство $g \circ f : A \rightarrowtail C \ \mathrm{Dom}(g \circ f) = A$
? $\mathrm{Range}(g \circ f) = C$?
$(\forall z \in C)(\exists y \in B)(g(y) = z)$
$(\forall y \in B)(\exists x \in A)(f(x) = y)$
$(\forall z \in C)(\exists x \in A)(g(f(x)) = z)$
$(\forall z \in C)\ f^{-1}(g^{-1}(z))\ f^{-1} \circ g^{-1}$

$\forall A, \overline{\overline{A}} = \overline{\overline{A}}$

Докзателство $\mathrm{id_A} : A \twoheadrightarrow A$

Ако $\overline{\overline{A}} = \overline{\overline{B}}$, то $\overline{\overline{B}} = \overline{\overline{A}}$

Доказателство $\exists f : A \twoheadrightarrow B,\ f^{-1} : B \twoheadrightarrow A$

Ако $\overline{\overline{A}} = \overline{\overline{B}}$ И $\overline{\overline{B}} = \overline{\overline{C}}$, то $\overline{\overline{A}} = \overline{\overline{C}}$

Доказателство $\exists f : A \twoheadrightarrow B$ и $g :\twoheadrightarrow C,\ g \circ f : A \twoheadrightarrow C$

$\overline{\overline{A}} \leq \overline{\overline{A}}$
Доказателство $\mathrm{id_A} : A \rightarrowtail A$

Ако $\overline{\overline{A}} \leq \overline{\overline{B}}$ И $\overline{\overline{B}} \leq \overline{\overline{C}}$, то $\overline{\overline{A}} \leq \overline{\overline{C}}$

Доказателство $\exists f : A \rightarrowtail B$ и $g : B \rightarrowtail C,\ g \circ f : A \rightarrowtail C.$

? $\overline{\overline{A}} < \overline{\overline{\overline{\mathcal{P}(A)}}}$?

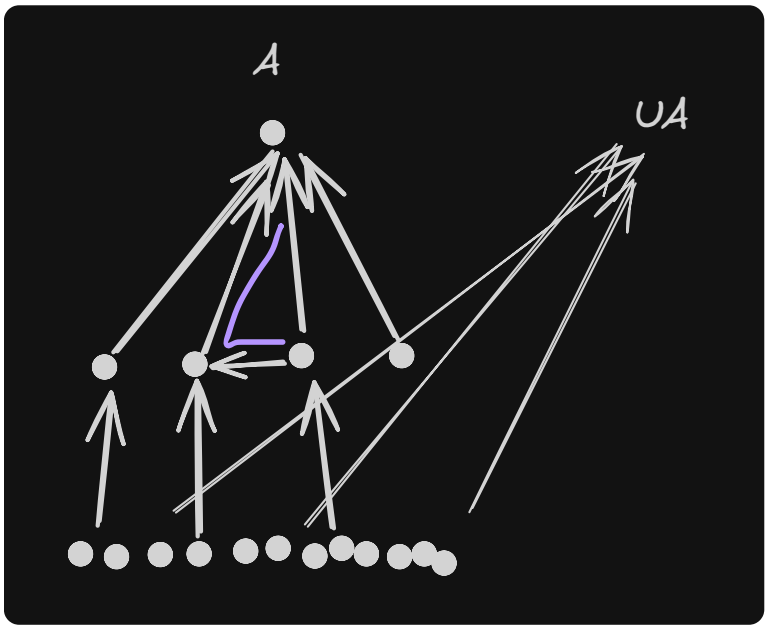
Вече сме доказали, че $\overline{\overline{A}} \neq \overline{\overline{\overline{\mathcal{P}(A)}}}$
? $\overline{\overline{A}} \leq \overline{\overline{\overline{\mathcal{P}(A)}}}$?
$f \leftrightharpoons \{p \mid p \in A \times \mathcal{P}(A) \ \& \ (\exists x \in A)(p = \langle x, \{x\}\rangle)\}$
$\mathrm{Range}(f) \subseteq \mathcal{P}(A)$
$\mathrm{Dom}(f) \subseteq A$
Let $x \in A \rightarrow \{x\} \in \mathcal{P}(A)$
$\Rightarrow \langle x, \{x\}\rangle \in f$
$\Rightarrow A \subseteq \mathrm{Dom}(f)$
$\Rightarrow A = \mathrm{Dom}(f)$
Let $\langle x, y\rangle \in f$ и $\langle x, y'\rangle \in f$
$y = \{x\} \ \& \ y' = \{x\} \Rightarrow y = y'$
$\Rightarrow f : A \rightarrow \mathcal{P}$
Let $\langle x, y\rangle \in f \ \& \ \langle x', y\rangle \in f$

$y = \{x\}\ \&\ y = \{x'\} \Rightarrow x = x'$
$\Rightarrow f : A \rightarrowtail \mathcal{P}(A)$
$\Rightarrow \overline{\overline{A}} \leq \overline{\overline{\mathcal{P}(A)}}\ \&\ \overline{\overline{A}} \neq \overline{\overline{\mathcal{P}(A)}}$
$\Rightarrow \overline{\overline{A}} < \overline{\overline{\mathcal{P}(A)}}$

Отново обединение?

$\cup A$
$x \in \cup A \leftrightarrow (\exists y \in A)(x \in y)$
Възможно ли е $x \in A$ и $x \in \cup A$?

Лилавия път е това



Твърдение $(\forall a \in A)(a \subseteq \cup A)$
Нека $a \in A$. ? $a \subseteq \cup A$?
Нека $x \in a$, $x \in \cup A$.

Нещо като горна граница ^ и то най-малката

Ако M : $(\forall a \in A)(a \subseteq M)$
? $\cup A \subseteq M$?
Нека $x \in \cup A$. Съществува $a \in A : x \in a$, $a \subseteq \cup M$, $x \in M$

Дефиниция Ако $\mathrm{Func}(f)$ и $f(x) = x$, x се нарича *неподвижна точка* на f (fixed point).
$f(f(f(x))) = x$

Теорема (Кнастер — Тарски)

Нека A е множествено и $f : \mathcal{P}(A) \to \mathcal{P}(A)$ и $(\forall X \in \mathcal{P}(A))(\forall Y \in \mathcal{P}(A))(X \subseteq Y \to f(X) \subseteq f(Y))$. Тогава f има неподвижна точка (и то най-голяма по величина). Съществува $X_0 \subseteq A$, такова че $f(X_0) = X_0$
Доказателство Нека $I \leftrightharpoons \{X \mid X \in \mathcal{P}(A)\ \&\ X \subseteq f(X)\}$
Знаем, че $\varnothing \in I$, значи $I \neq \varnothing$. Нека $X_0 \in I$.

? $f(X_0) \in I$? $X_0 \subseteq f(X_0)$
? $f(X_0) \subseteq f(f(X_0))$? — по монотонност
Търсим $f(X) = X$.
Да разгледаме $\cup I$, нещо като горна граница за тези точки.
? $\cup I \subseteq A$?
Нека $x \in \cup I$. Съществува $y \in I : x \in y$.
$y \in \mathcal{P}(A)$, значи $y \subseteq A$. $x \in A$
$\cup I \in \mathcal{P}(A)$
? $\cup I \in I$?
? $\cup I \subseteq f(\cup I)$?
Нека $x \in \cup I$. Значи съществува $\boxed{y \in I}$, $\boxed{x \in y}$. $y \in I$, Значи $y \subseteq \cup I$ (свойство от по-горе) и $\boxed{y \subseteq f(y)}$.
От оградените казваме??? $f(y) \subseteq f(\cup I)$
$x \in y \subseteq f(y) \subseteq f(\cup I)$
$x \in f(\cup I)$. $\cup I \subseteq f(\cup I)$, $\cup I \in I$.
? $f(\cup I) \subseteq \cup I$?

$f(\cup I) \in I$, $f(\cup I) \subseteq \cup I$

$\cup I \subseteq f(\cup I)$
$\boxed{f(\cup I)} \subseteq f(f(\cup I))$

Нека X_0 е неподвижна точка $f(X_0) = X_0$
$X_0 \subseteq f(X_0)$
$X_0 \in I$. $X_0 \subseteq \cup I$

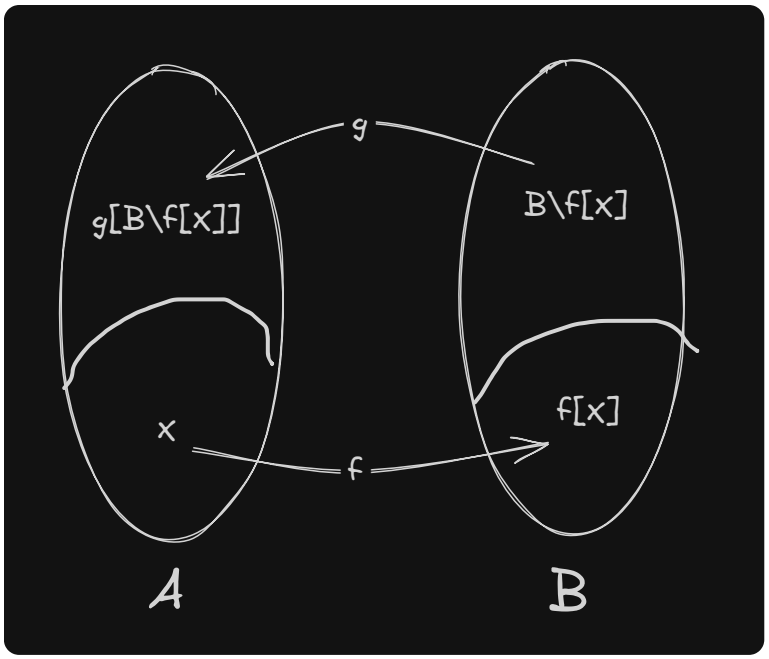
$\cup I \in I$. Примерно $\{\{x\}, \{y\}, \{x, y\}\}$.

Домашно 1)

Твърдение Ако $X \subseteq Y \subseteq A$ и $f : A \to A$, то $f[X] \subseteq f[Y]$
Доказателство Нека $a \in f[X]$. Съществува $x \in X$, $f(x) = a$. $x \in Y$. $a \in f[Y]$

Теорема (Кантор — Шрьодер — Бернщайн)

Ако $\overline{\overline{A}} \leq \overline{\overline{B}}$ и $\overline{\overline{B}} \leq \overline{\overline{A}}$, то $\overline{\overline{A}} = \overline{\overline{B}}$.



Доказателство Нека $F(X) \leftrightharpoons A \setminus \left(g\left[B \setminus f[X]\right]\right)$

$F : \mathcal{P}(A) \to \mathcal{P}(A)$.

$F \leftrightharpoons \left\{p \mid p \in \mathcal{P}(A) \times \mathcal{P}(A) \ \& \ \exists x \exists y \left(p = \langle x, y\rangle \ \& \ y = A \setminus g\left[B \setminus f[X]\right]\right)\right\}$

Нека $a \subseteq A$. $\langle a, g\left[B \setminus f[a]\right]\rangle \in F$. $a \in \mathrm{Dom}(F)$.

Нека $x \in \mathcal{P}(A)$ и $y, y' \in \mathcal{P}(A)$.

$\langle x, y\rangle \in F$ и $\langle x, y'\rangle \in F$

$y = A \setminus g\left[B \setminus f[x]\right]$

$y' = A \setminus g\left[B \setminus f[x]\right]$

$y = y'$

$F : \mathcal{P}(A) \to \mathcal{P}(A)$.

Ако $X \subseteq Y \subseteq A$, то ? $F(X) \subseteq F(Y)$?

Нека $X \subseteq Y (\subseteq A)$.

$f[X] \subseteq f[Y]$

$B \setminus f[Y] \subseteq B \setminus f[X]$

$b \in B \setminus f[Y]$

$b \in B$ и $b \notin f[Y]$, $b \notin f[X]$.

$g\left[B \setminus f[Y]\right] \subseteq g\left[B \setminus f[X]\right]$

$A \setminus g\left[B \setminus f[X]\right] \subseteq A \setminus g\left[B \setminus f[Y]\right]$

$a \in A \setminus g\left[B \setminus f[X]\right]$, $a \in A$ и $a \notin g\left[B \setminus f[X]\right]$

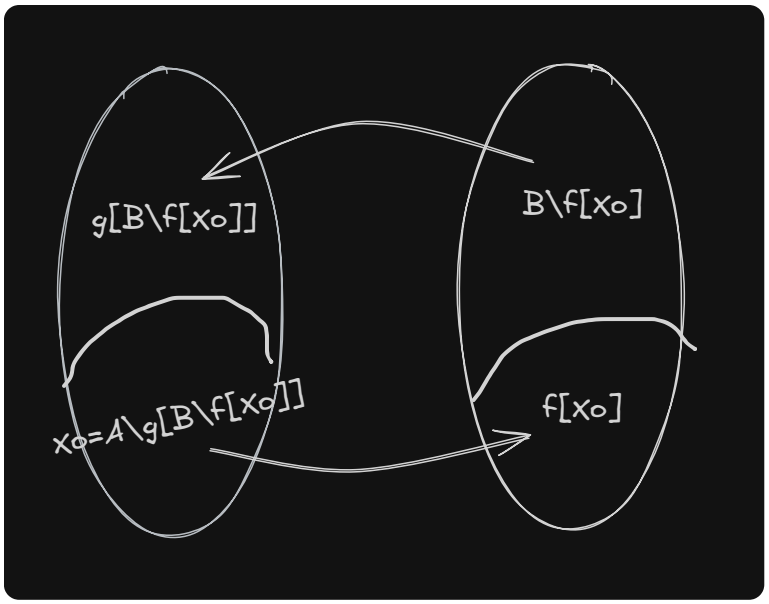
$a \notin g\left[B \setminus f[Y]\right]$

От горното и $a \in A$: $a \in A \setminus g\left[B \setminus f[Y]\right]$.

Значи f е монотонно и има неподвижна точка,

по **Кнастър-Тарски** съществува $X_0 \in \mathcal{P}(A)$ $F(X_0) = X_0$.

$A \setminus g\left[B \setminus f[X_0]\right] = X_0$



? $(f \restriction X_0) \cup (g^{-1} \restriction (A \setminus X_0))$?

? $A \rightarrowtail\!\!\!\!\!\rightarrow B$?

$f \restriction X_0 : X_0 \rightarrowtail\!\!\!\!\!\rightarrow f[X_0]$ (защото $f : A \rightarrowtail B$ и $X_0 \subseteq A$ и $f[X_0] \subseteq B$)
$f \restriction X_0 : X_0 \to B$
$f \restriction X_0 = \Big(X_0 \times \mathrm{Range}(f)\Big) \cap f$
$f \restriction X_0 \subseteq X_0 \times \mathrm{Range}(f)$
? $\mathrm{Dom}(f \restriction X_0) = X_0$?

Нека $x \in X_0$. $\langle x, f(x)\rangle \in f$

$x \in X_0$, $f(x) \in \mathrm{Range}(f)$
$\langle x, f(x)\rangle \in X_0 \times \mathrm{Range}(f)$
$\langle x, f(x)\rangle \in (X_0 \times \mathrm{Range}(f)) \cap f = f \restriction X_0$
Нека $\langle x, y\rangle \in f \restriction X_0$ и $\langle x, y'\rangle \in f \restriction X_0$

$\langle x, y\rangle \in f$ и $\langle x, y'\rangle \in f$. $y = y'$

$\langle x', y\rangle \in f \restriction X_0$ и $\langle x'', y\rangle \in f \restriction X_0$.

$\langle x', y\rangle \in f$ $\langle x'', y\rangle \in f$
$x' = x''$
$f \restriction X_0 : X_0 \rightarrowtail B$
? $\mathrm{Range}(f \restriction X_0) = f[X_0]$?

LHS: $y \in \mathrm{Range}(f \restriction X_0)$

$\exists x \Big(\langle x, y\rangle \in f \ \&\ \langle x, y\rangle \in X_0 \times \mathrm{Range}\, f)\Big)$

$y = f(x)$ и $x \in X_0$

RHS: $y \in f[X_0] \leftrightarrow (\exists x \in X_0)(y = f(x))$

Да.
$f \restriction X_0 : X_0 \rightarrowtail\!\!\!\!\!\rightarrow f[X_0]$
По същото съображение, $g \restriction \Big(B \setminus f[X_0]\Big) : B \setminus f[X_0] \rightarrowtail\!\!\!\!\!\rightarrow g\Big[B \setminus f[X_0]\Big]$

$\Big(g \restriction (B \setminus f[X_0])\Big)^{-1} : g\Big[B \setminus f[X_0]\Big] \rightarrowtail\!\!\!\!\!\rightarrow B \setminus f[X_0]$
Какво знаем за неподвижната точка X_0?

$A \setminus g\Big[B \setminus f[X_0]\Big] = X_0$

$A \setminus X_0 = g\Big[B \setminus f[X_0]\Big]$

$\Big(g \restriction (B \setminus f[X_0])\Big)^{-1} : A \setminus X_0 \to B \setminus f[X_0]$

? $X_0 \cap A \setminus X_0 = \varnothing$? Да

? $f[X_0] \cap (B \setminus f[X_0]) = \varnothing$? Също да

Имаме биекция от лявото към лявото и от дясното към дясното, съответно по твърдение 1) от домашното имаме $(f \restriction X_0) \cup \left(g \restriction (B \setminus f[X_0])\right)^{-1} : X_0 \cup (A \setminus X_0) \rightarrowtail\!\!\!\!\!\twoheadrightarrow f[X_0] \cup (B \setminus f[X_0])$
Обаче това е $(f \restriction X_0) \cup \left(g \restriction (B \setminus f[X_0])\right)^{-1} : A \rightarrowtail\!\!\!\!\!\twoheadrightarrow B$, тоест $\overline{\overline{A}} = \overline{\overline{B}}$.

Домашно

1. *Твърдение* Нека $A_1 \cap A_2 = \varnothing$, $B_1 \cap B_2 = \varnothing$.
   Нека $f : A_1 \rightarrowtail\!\!\!\!\!\twoheadrightarrow B_1$ и $g : A_2 \rightarrowtail\!\!\!\!\!\twoheadrightarrow B_2$
   ? $\overline{\overline{A_1 \cup A_2}} = \overline{\overline{B_1 \cup B_2}}$?
   $f \cup g : A_1 \cup A_2 \rightarrowtail\!\!\!\!\!\twoheadrightarrow B_1 \cup B_2$
2. *Твърдение* Ако $\overline{\overline{A}} < \overline{\overline{B}}$ и $\overline{\overline{B}} < \overline{\overline{C}}$, то $\overline{\overline{A}} < \overline{\overline{C}}$
3. *Твърдение* $A \subseteq B \subseteq C$ и $\overline{\overline{A}} = \overline{\overline{C}}$, то $\overline{\overline{A}} = \overline{\overline{B}}$